Bouncing Ball System

AMATH 575 Final Project

Christine Lind

Department of Applied Mathematics
University of Washington

June 2, 2005

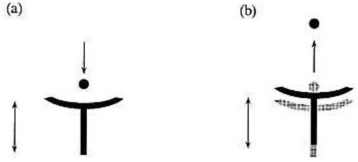
Outline

Bouncing Ball System Description

Simple Physical System

Interaction between:

- Ball
- Sinusoidally Oscillating Table



Bouncing Ball System Description

Initial Assumptions:

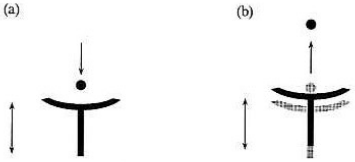
(x_k, t_k) - ball position and time of k^{th} impact

- Between Impacts, ball obeys Newton's Laws:

$$x(t) = x_k + v_k(t - t_k) - \frac{g}{2}(t - t_k)^2 \qquad t_k \leq t \leq t_{k+1}$$

- Table is unaffected by impacts:

$$s(t) = A\left(\sin(\omega t + \theta_0) + 1\right)$$



Bouncing Ball System Description

Solve for Next Impact Time

$d(t) = x(t) - s(t)$ - distance between ball and table

- First $t > t_k$ where $d(t) = 0$ is t_{k+1}, next impact time!

$$\begin{aligned} 0 = d(t_{k+1}) &= x_k + v_k(t_{k+1} - t_k) - \frac{g}{2}(t_{k+1} - t_k)^2 \\ &- A\left(\sin(\omega t_{k+1} + \theta_0) + 1\right) \end{aligned}$$

- Note that at time t_k:

$$x_k = s(t_k) = A\left(\sin(\omega t_k + \theta_0) + 1\right)$$

Bouncing Ball System Description

Solve for Next Impact Time

$d(t) = x(t) - s(t)$ - distance between ball and table

- Then the (Implicit) Time-Equation is:

$$\begin{aligned} 0 &= A\sin(\omega t_k + \theta_0) + v_k(t_{k+1} - t_k) - \frac{g}{2}(t_{k+1} - t_k)^2 \\ &- A\sin(\omega t_{k+1} + \theta_0) \end{aligned}$$

- Note that v_k is still unknown
- Find Velocity-Equation

Bouncing Ball System Description

Solve for Impact Velocity

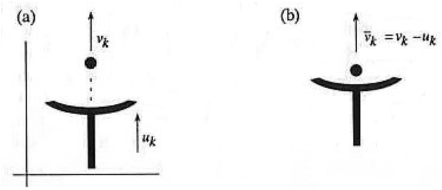
First look at two different frames of reference

(a) Ground (Lab) Frame of Reference:

- v_k - ball velocity at impact k
- u_k - table velocity at impact k

(b) Table Frame of Reference:

- $\overline{v}_k = v_k - u_k$ - ball velocity at impact k



Bouncing Ball System Description

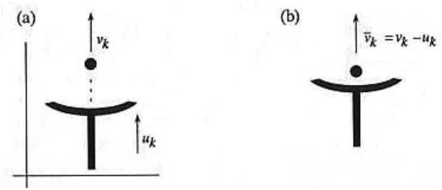
Solve for Impact Velocity

$\overline{v}'_k$ - velocity just before impact k

$\overline{v}_k$ - velocity just after impact k

α - coefficient of restitution (describes damping)

- $\overline{v}_k = -\alpha \overline{v}'_k$
- $0 \leq \alpha \leq 1$
  - $\alpha = 1$ - no energy loss (no damping - elastic collision)
- Transform back to Ground (Lab) Reference Frame...



Bouncing Ball System Description

Solve for Impact Velocity

$v_{k+1} = (1+\alpha)u_{k+1} - \alpha v'_{k+1}$

- Recall that for $t_k \le t \le t_{k+1}$ the ball position is described by:

$$\begin{aligned} x(t) &= x_k + v_k(t - t_k) - \frac{g}{2}(t - t_k)^2 \\ v'_{k+1} &= x'(t_{k+1}) = v_k - g(t_{k+1} - t_k) \end{aligned}$$

- The table position is given by:

$$\begin{aligned} s(t) &= A\left(\sin(\omega t + \theta_0) + 1\right) \\ u_{k+1} &= s(t_{k+1}) = A\omega \cos(\omega t_{k+1} + \theta_0) \end{aligned}$$

- Then we can solve for the Impact Velocity

Bouncing Ball System Description

Solve for Impact Velocity

- Impact Velocity Equation:

$$\begin{aligned} v_{k+1} &= (1+\alpha)A\omega\cos(\omega t_{k+1} + \theta_0) \\ &- \alpha\,(v_k - g(t_{k+1} - t_k)) \end{aligned}$$

Bouncing Ball Exact Equations

System is described by

- Time Equation:

$$\begin{aligned} 0 &= A\sin(\omega t_k + \theta_0) + v_k(t_{k+1} - t_k) \\ &- \frac{g}{2}(t_{k+1} - t_k)^2 - A\sin(\omega t_{k+1} + \theta_0) \end{aligned}$$

- Velocity Equation:

$$\begin{aligned} v_{k+1} &= (1+\alpha)A\omega\cos(\omega t_{k+1} + \theta_0) \\ &- \alpha\left(v_k - g(t_{k+1} - t_k)\right) \end{aligned}$$

Bouncing Ball Exact Equations

Non-Dimensionalization!

Too many parameters to study the system efficiently

Parameters - α, A, ω, g

- Transform system into dimensionless variables:

$$\begin{aligned} \theta_k &= \omega t_k + \theta_0 \\ \nu_k &= \frac{2\omega}{g} v_k \end{aligned}$$

- New Parameter

$$\beta = \frac{2\omega^2(1+\alpha)A}{g}$$

Bouncing Ball Exact Equations

Dimensionless System is described by

- Phase Equation:

$$\begin{aligned} 0 &= \beta\left(\sin\theta_k - \sin\theta_{k+1}\right) \\ &+ (1+\alpha)\left(\nu_k(\theta_{k+1}-\theta_k) - (\theta_{k+1}-\theta_k)^2\right) \end{aligned}$$

- Velocity Equation:

$$\nu_{k+1} = \beta\cos\theta_{k+1} - \alpha\left(\nu_k - 2(\theta_{k+1}-\theta_k)\right)$$

- Now we can study the system simply by varying α and β.

Bouncing Ball Exact Equations

Dimensionless System is described by

- Phase Equation:

$$\begin{aligned} 0 &= \beta\,(\sin\theta_k - \sin\theta_{k+1}) \\ &+ (1+\alpha)\left(\nu_k(\theta_{k+1}-\theta_k) - (\theta_{k+1}-\theta_k)^2\right) \end{aligned}$$

- Velocity Equation:

$$\nu_{k+1} = \beta\cos\theta_{k+1} - \alpha\,(\nu_k - 2(\theta_{k+1}-\theta_k))$$

- Implicit Maps can be hard to analyze & simulate $\Rightarrow$ make an approximation that will give us an Explicit Map...

Bouncing Ball Approximation

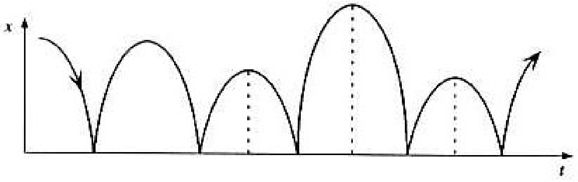
High Bounce Approximation

Assume:

change in table height $\ll$ maximum height of the ball

- Ball orbit symmetric about the maximum height:

$$x_k = x_{k+1} \qquad v'_{k+1} = -v_k$$

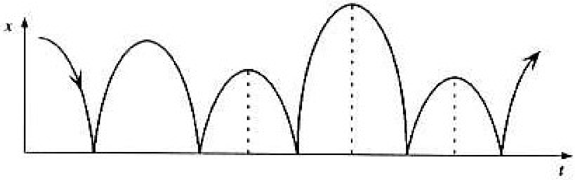


Bouncing Ball Approximation

High Bounce Approximation

$v'_{k+1} = -v_k$

- Recall:
$$v'_{k+1} = v_k - g(t_{k+1} - t_k) = -v_k$$
- Explicit Time Map:
$$t_{k+1} = t_k + \frac{2}{g} v_k$$
- Use equation above to solve for the velocity map, and non-dimensionalize...



Bouncing Ball Approximation

High Bounce Equations

- Phase Equation:

$$\theta_{k+1} = \theta_k + \nu_k \qquad (\text{mod } 2\pi)$$

- Velocity Equation:

$$\nu_{k+1} = \alpha \nu_k + \beta \cos(\theta_k + \nu_k)$$

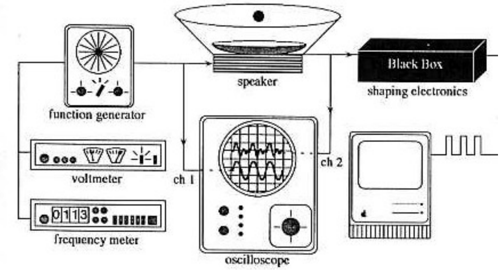
- For $\alpha = 1$, this is the “Standard” Map!

Outline

Bouncing Ball Experiment

Speakers and Function Generators

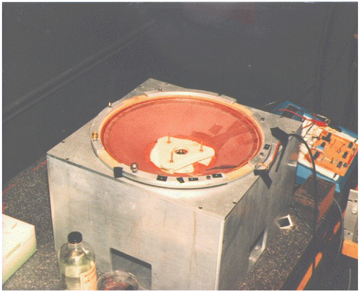
The physical system can be explored using a setup similar to the schematic shown below.



Bouncing Ball Experiment

Experimental Set-up

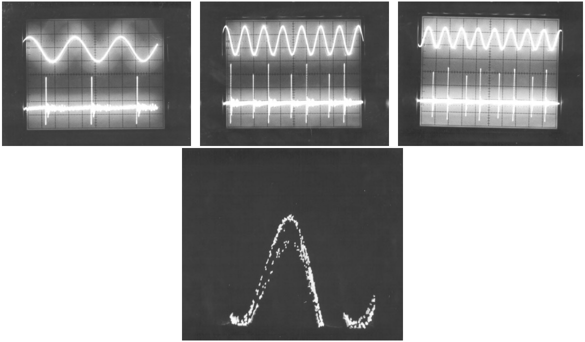
Nicholas B. Tufillaro's experimental set-up at Bryn Mawr College (circa 1985).



Bouncing Ball Experiment

Experimental Results

Nicholas B. Tufillaro's experimental results, (see references)

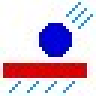


Bouncing Ball Program

Bouncing Ball Simulation Program

Nicholas B. Tufillaro wrote a program called Bouncing Ball for the Apple Macintosh. Bouncing Ball simulates experiments by numerically solving the exact equations for the system.
(You can download this program from his website)

- Bisection Method used (NOT Newton's Method) because of ease of coding and stability

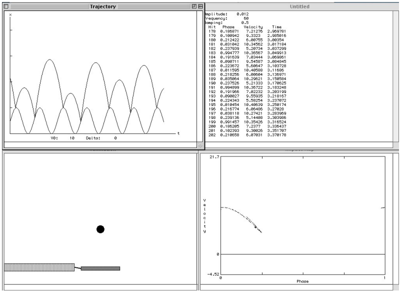


Bouncing Ball Program

Bouncing Ball Simulations

default settings show four different windows:

- Trajectory
- Impact Data
- Animation
- Impact Map

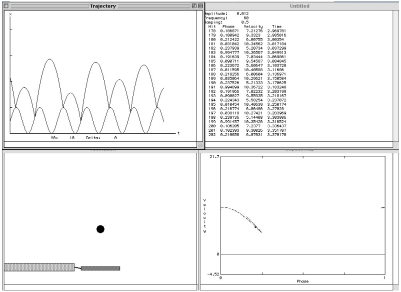


Bouncing Ball Program

Bouncing Ball Simulations

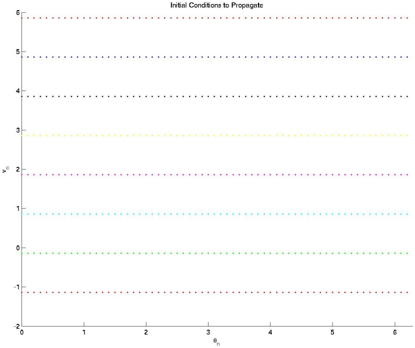
Program can also plot

- Bifurcation Diagrams
- Basins of Attraction - for periodic points of period 1,2,3,4,8
- ...and play sounds at impact events - hear chaos!



Matlab Simulations

It is easy to iterate the High Bounce Approximation, or "Standard" Map in Matlab.
Initial conditions propagated in the following Matlab figures are shown below:



Bouncing Ball Matlab Simulations

Matlab Simulations - High Bounce Approximation

Recall that the map in question is given by:

$$\theta_{k+1} = \theta_k + \nu_k \qquad (\text{mod } 2\pi)$$

$$\nu_{k+1} = \alpha\nu_k + \beta\cos(\theta_k + \nu_k)$$

This map has fixed points (θ, ν):

- $\left(\pm\arccos\left(\frac{2k\pi(1-\alpha)}{\beta}\right), 2k\pi\right)$
- for integer values of k

Bouncing Ball Matlab Simulations

Matlab Simulations - High Bounce Approximation

For $\alpha = 1$ we get exactly the "Standard" Map, and

$$\theta_{k+1} = \theta_k + \nu_k \qquad \pmod{2\pi}$$

$$\nu_{k+1} = \nu_k + \beta \cos(\theta_k + \nu_k)$$

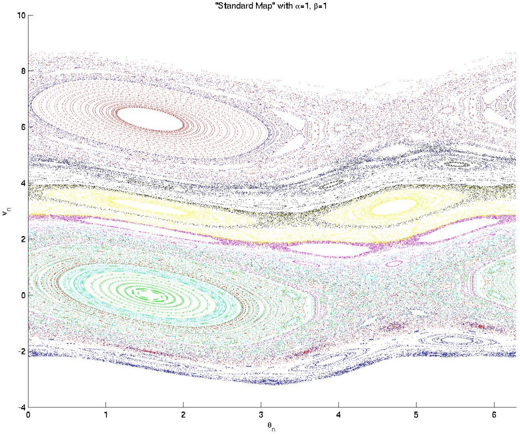
fixed points (θ, ν):

- $\left(\frac{\pi}{2}, 2k\pi\right) \qquad \left(\frac{3\pi}{2}, 2k\pi\right)$
- for integer values of k

Bouncing Ball Matlab Simulations

High Bounce Approximation - $\alpha = 1$, $\beta = 1$

(This is the Standard Map)



Bouncing Ball Matlab Simulations

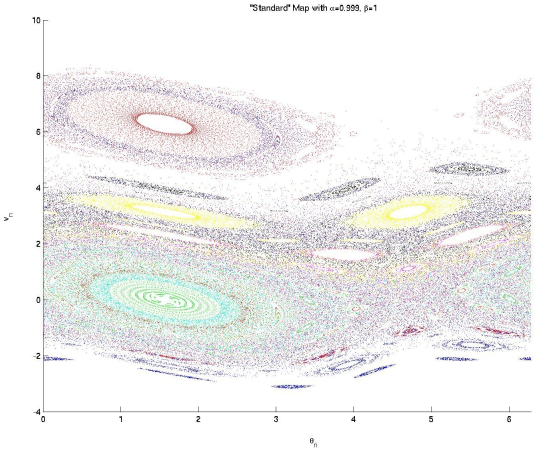
High Bounce Approximation

What will happen as we turn on the dissipation in the system?

- Center at $\left(\frac{\pi}{2}, 0\right)$ becomes Stable
- Centers at $\left(\frac{\pi}{2}, 2k\pi\right)$ shift and become Stable (for α not "too small")

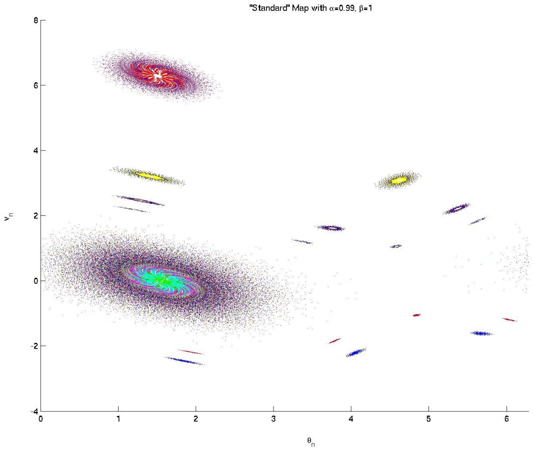
Bouncing Ball Matlab Simulations

High Bounce Approximation - $\alpha = .999$, $\beta = 1$



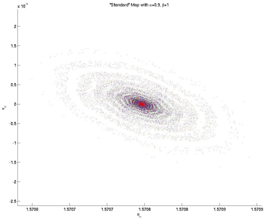
Bouncing Ball Matlab Simulations

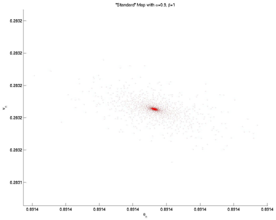
High Bounce Approximation - $\alpha = .99$, $\beta = 1$



Bouncing Ball Matlab Simulations

High Bounce Approximation - $\alpha = .9$, $\beta = 1$





Outline

Bouncing Ball Simulations

High Bounce - Exact System Comparison

Visually compare Standard Map $\alpha = 1$, $\beta = 1$ to Bouncing Ball Simulation with

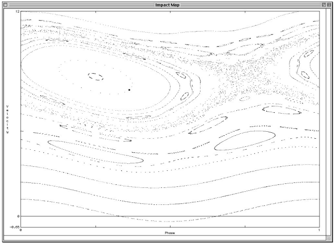
- f = 60 Hz A = 0.00172563 $\alpha = 1$
- $\beta = \frac{2\omega^2(1+\alpha)A}{g} = \frac{2(2\pi*60)^2(1+1)*0.00172563}{981} \approx 1$

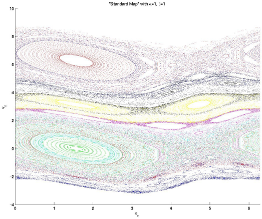
Then compare the Matlab Simulation with $\alpha = 0.9$, $\beta = 1$ with Bouncing Ball and

- f = 60 Hz A = 0.00181645 $\alpha = 0.9$
- $\beta = \frac{2\omega^2(1+\alpha)A}{g} = \frac{2(2\pi*60)^2(1+0.9)*0.00181645}{981} \approx 1$

Bouncing Ball Simulations

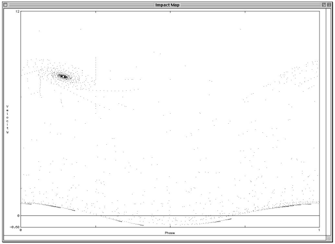
High Bounce - Exact System Comparison $\alpha = 1$, $\beta = 1$

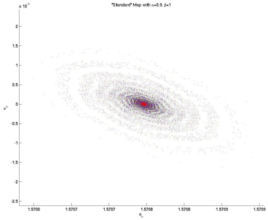




Bouncing Ball Simulations

High Bounce - Exact System Comparison $\alpha = 0.9$, $\beta = 1$



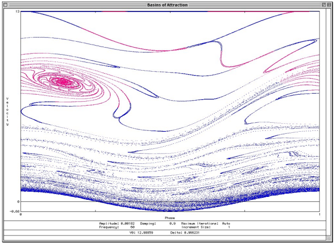


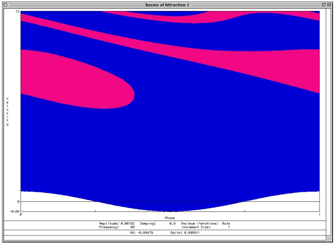
Bouncing Ball Simulations

Exact System Basin of Attraction $\alpha = 0.9$, $\beta = 1$

blue points get stuck to the table

- High Bounce Approx. is invertible $\Rightarrow$ cannot capture this behavior

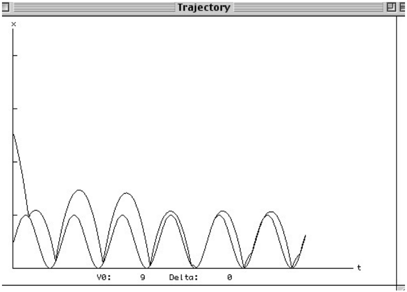




Bouncing Ball Simulations

High Bounce - Exact System Comparison

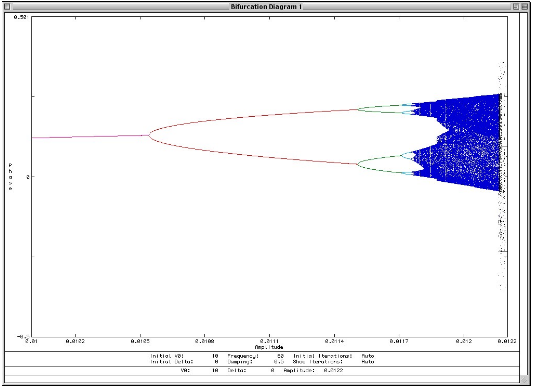
High Bounce Approx. also cannot describe situations where the ball rests on the table, but eventually leaves the table again - "Sticking Solutions"



High Bounce - Exact System Comparison

Bifurcation

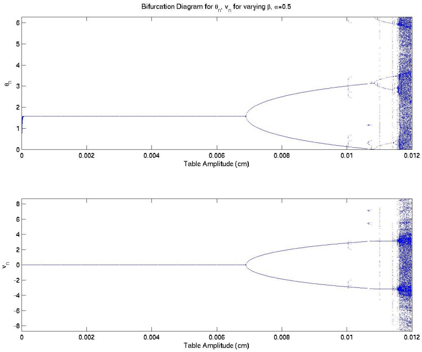
Exact System exhibits the classic “Period-Doubling” route to chaos for $\alpha = 0.5$:



High Bounce - Exact System Comparison

Bifurcation

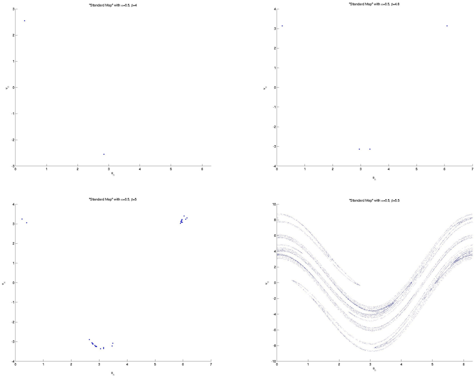
High Bounce Approx. reproduces "Period-Doubling" route to chaos for $\alpha = 0.5$



High Bounce - Exact System Comparison

Bifurcation

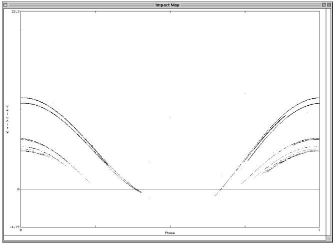
High Bounce Approx. reproduces "Period-Doubling" route to chaos for $\alpha = 0.5$

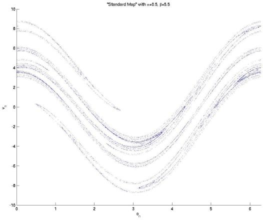


High Bounce - Exact System Comparison

Strange Attractor

Both models are shown for $\alpha = 0.5$, $\beta = 5.5$





Outline

Bouncing Ball System Conclusions

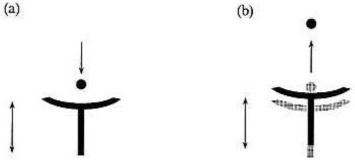
Simple Physical system ⇒ Chaos

Interaction between:

- Ball
- Sinusoidally Oscillating Table

Leads to chaotic behavior for certain parameter values

- Experimental set-up can allow physical measurement of Feigenbaum's constant



Bouncing Ball System Conclusions

High Bounce - Exact System Comparison

Models have good qualitative agreement overall

- Exact System
  - Implicit Equations - Hard to Solve/Simulate
  - Can describe "sticking solutions" (not invertible)
- High Bounce Approximation
  - Explicit Equations - Easy to Solve/Simulate
  - Invertible $\Rightarrow$ cannot describe "sticking solutions"
  - Can describe non-physical situations (ball below table)

Questions?

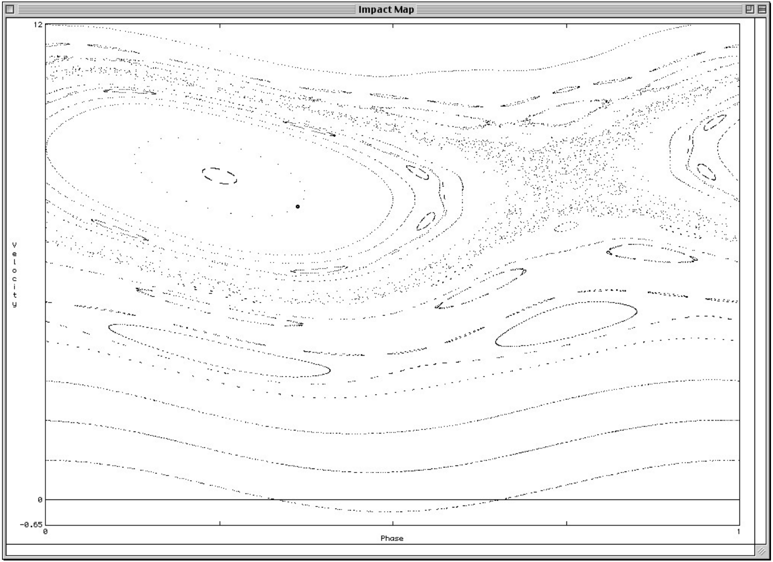
Bouncing Ball System - References

- S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos - 2nd Ed.* Springer-Verlag, 2003.
- N.B. Tufillaro, T. Abbott, and J. Reilly. *An Experimental Approach to Nonlinear Dynamics and Chaos,* Addison Wesley, 1992.
- T. M. Mello and N. B. Tufillaro. Strange attractors of a bouncing ball, *American Journal of Physics,* 55 (4), 316 (1987).
- N. B. Tufillaro and A. M. Albano. Chaotic dynamics of a bouncing ball, *American Journal of Physics,* 54 (10), 939 (1986).

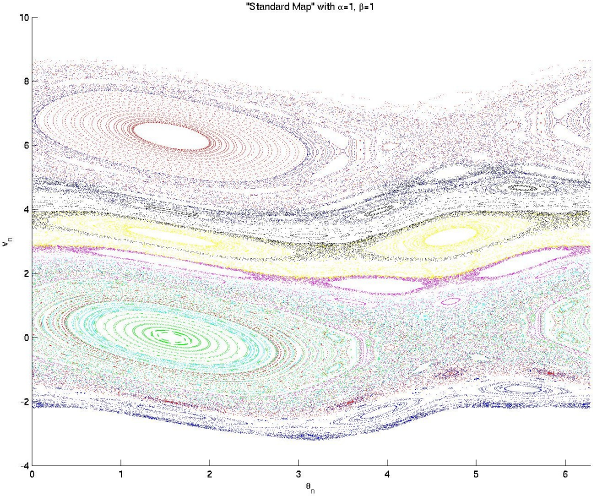
- N.B. Tufillaro's Website (Contains more references) http://www.drchaos.net/drchaos/bb.html

Outline

Bouncing Ball System - Extras

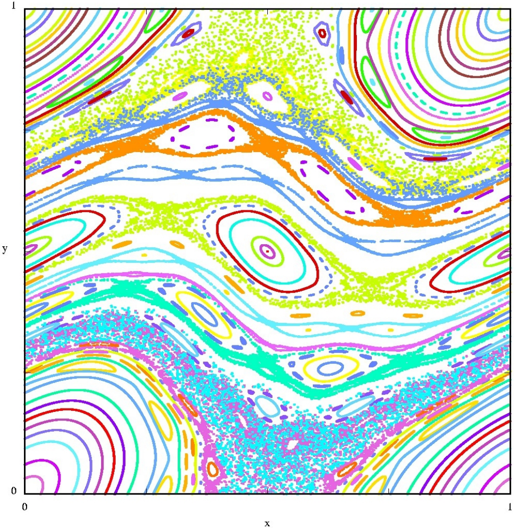


Bouncing Ball System - Extras

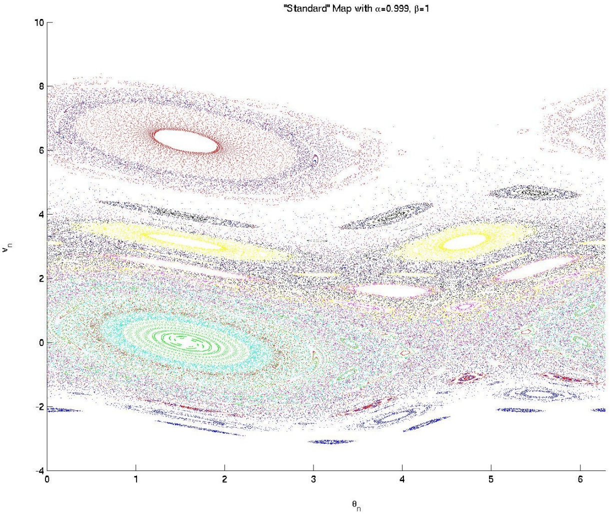


Bouncing Ball System - Extras

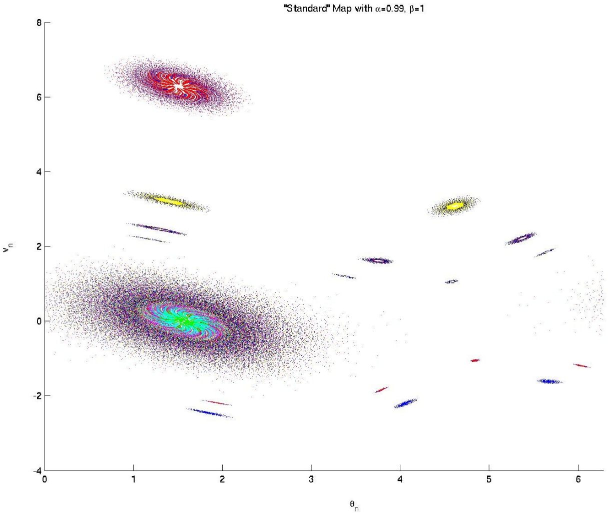
Standard Map



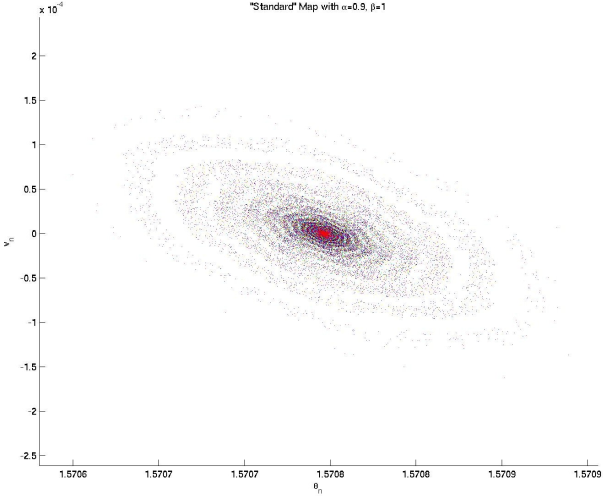
Bouncing Ball System - Extras



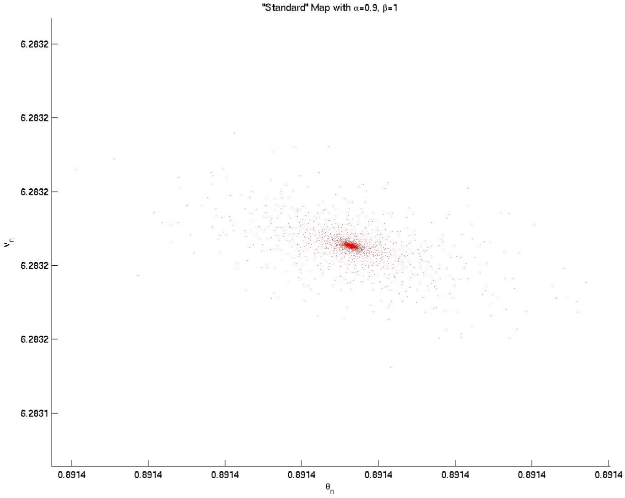
Bouncing Ball System - Extras



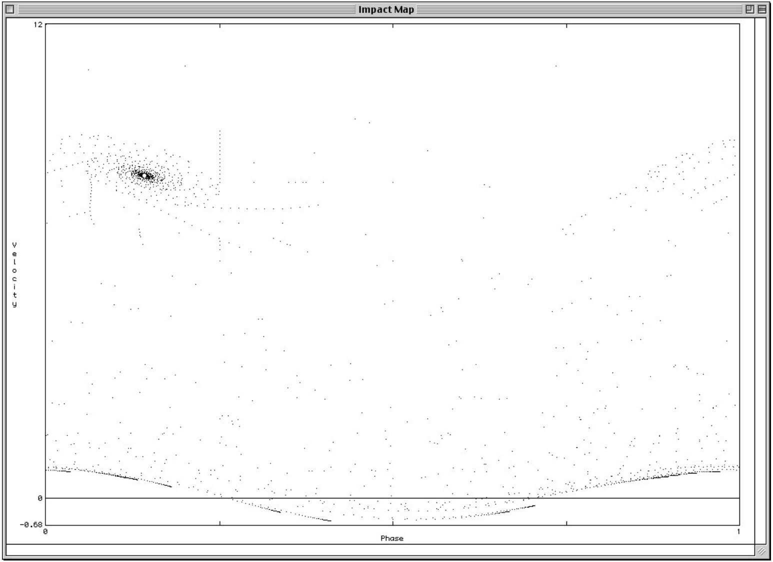
Bouncing Ball System - Extras



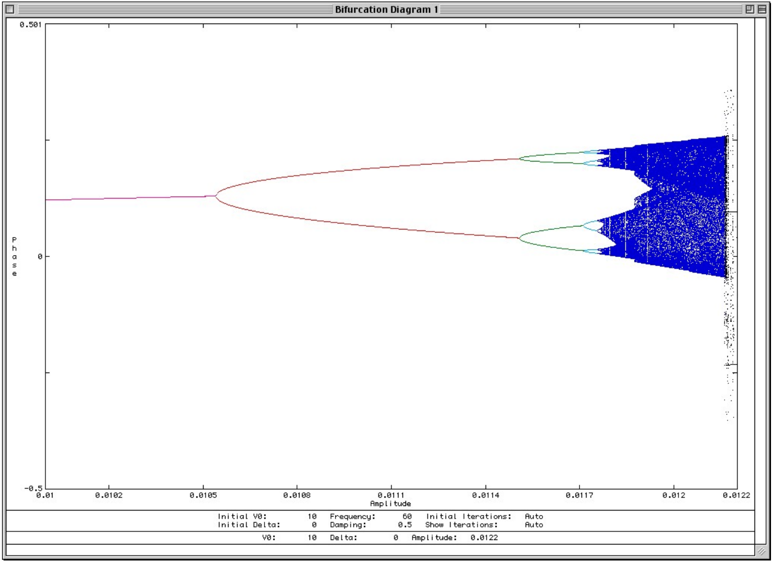
Bouncing Ball System - Extras



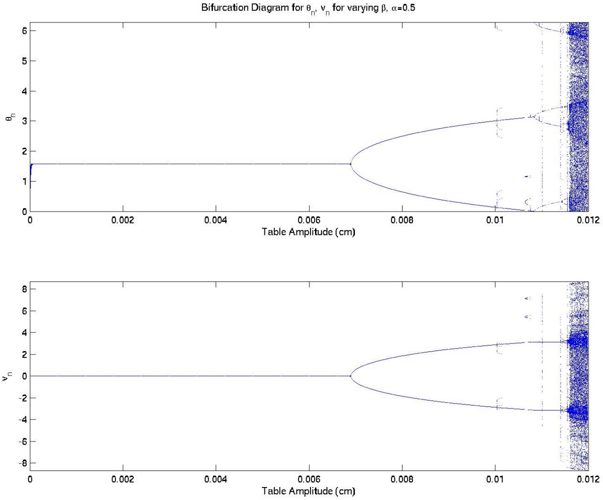
Bouncing Ball System - Extras



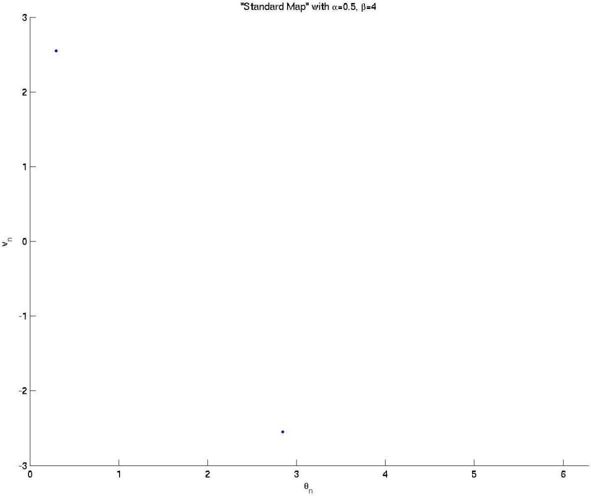
Bouncing Ball System - Extras



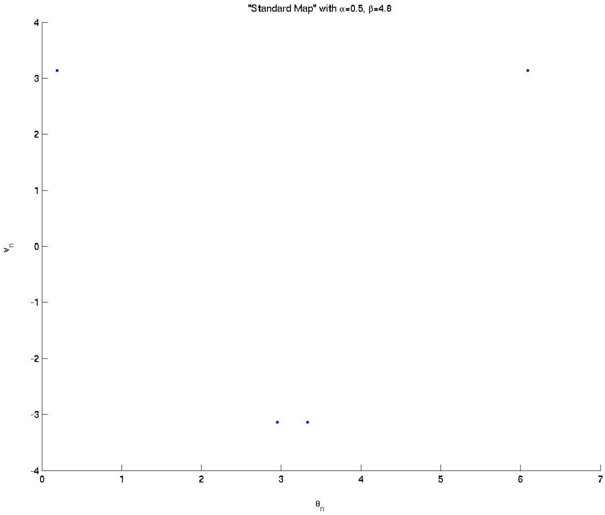
Bouncing Ball System - Extras



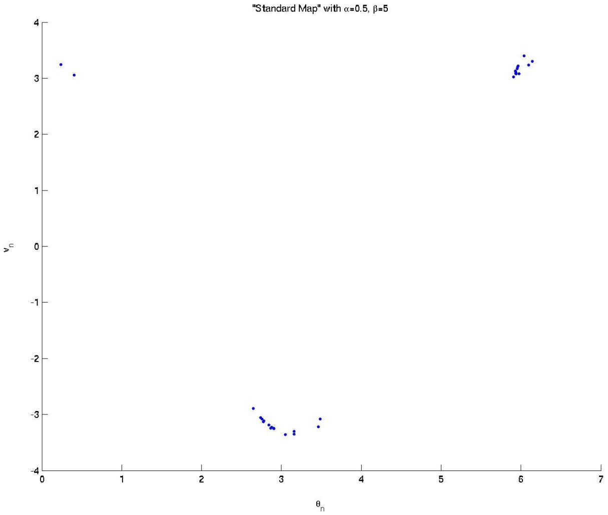
Bouncing Ball System - Extras



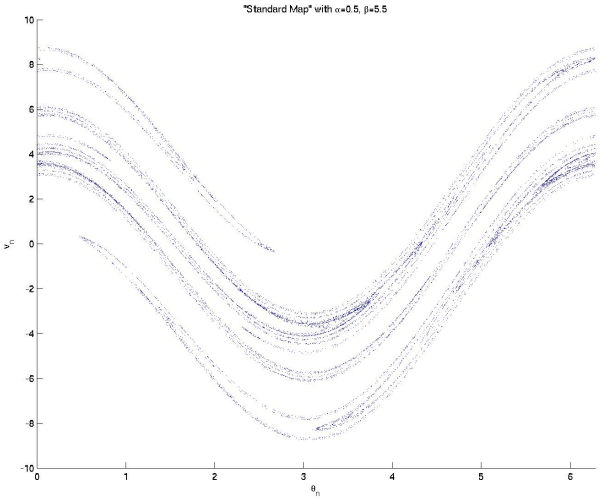
Bouncing Ball System - Extras



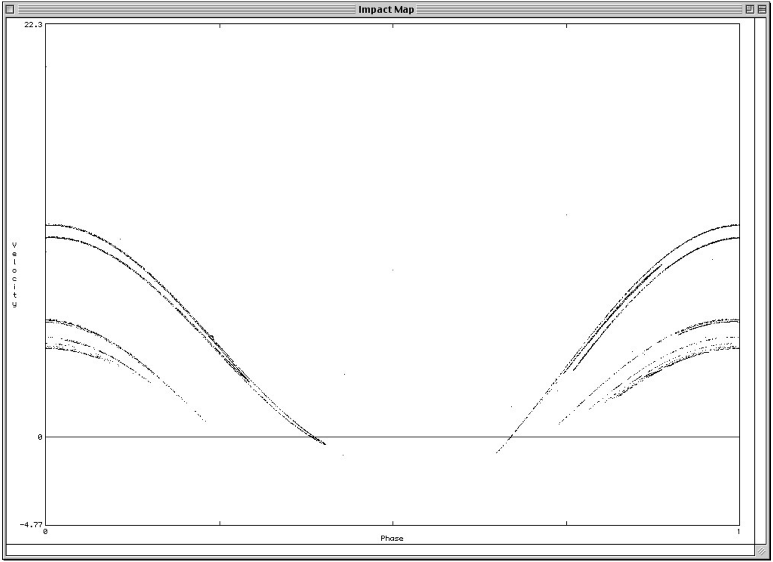
Bouncing Ball System - Extras



Bouncing Ball System - Extras



Bouncing Ball System - Extras



Bouncing Ball System - Extras

Feigenbaum's Delta

$$\delta = \lim_{n \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = 4.669202$$

λ is the value of A for which bifurcation occurs:

- $A_1 = \lambda_1 = 0.0106$, $A_2 = \lambda_2 = 0.0115$,
- $A_3 = \lambda_3 = 0.0117$

$$\delta \approx \frac{0.0115 - 0.0106}{0.0117 - 0.0115} = 4.5$$